Uniform attractors of non-autonomous dissipative semilinear wave equations*

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Received June 17, 2002; revised September 16, 2002

Abstract The asymptotic long time behaviors of a certain type of non-autonomous dissipative semilinear wave equations are studied. The existence of uniform attractors is proved and their upper bounds for both Hausdorff and Fractal dimensions of uniform are given when the external force satisfies suitable conditions.

Keywords: semilinear wave equations, uniform attractor, Hausdorff and Fractal dimensions.

In the past two decades, the asymptotic long time behaviors of autonomous dissipative wave equations were studied extensively[1~6]. Since the definition of uniform attractors was given by Haraux[7], most researchers agree that the asymptotic long time behaviors of non-autonomous dissipative evolution systems can be described by uniform attractors. Recently, Chepyzhov and Vishik studied asymptotic long time behaviors of non-autonomous dissipative dynamical systems[8,9]. But most parts of their papers dealt with parabolic equations, for example, the well-known Navier-Stokes equations and a certain type of reaction-diffusion equations. In this paper, we extended their results of the existence of uniform attractors to general dissipative semilinear wave equations and developed a method to estimate the upper bounds for Hausdorff and Fractal dimensions of uniform attractors of a certain type of dissipative wave equations, i. e the general non-autonomous semilinear wave equations as follows:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \alpha \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(x,t), \\ \quad \text{in} \quad \Omega \times [\tau, \infty) \\ u(x,\tau) = u_{0\tau}(x), \\ \dfrac{\partial u(x,\tau)}{\partial t} = u_{1\tau}(x), \\ u = 0, \quad \text{on} \quad \partial\Omega \times [\tau, \infty), \end{cases} \tag{1}$$

where $\alpha > 0$ is a constant, $\Omega \in R^n (n \geqslant 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\tau \in R$ is the initial time.

1 Preliminaries

First, let us introduce some notations and functional spaces as the following: $H^2(\Omega)$, $H_0^1(\Omega)$, $L^2(\Omega)$ are usual Sobolev spaces.

$$V = H_0^1(\Omega), \quad H = L^2(\Omega), \quad u' = \frac{\partial u}{\partial t}.$$
$$Au = -\Delta u, \quad D(A) = H^2(\Omega) \cap V,$$
$$E_0 = V \times H, \quad E_1 = D(A) \times V.$$

$V' = H^{-1}(\Omega)$ is the dual space of V, and $\langle \cdot, \cdot \rangle$ denotes the dual product between V and V',

$$((u,v)) = \langle Au, v \rangle,$$
$$\| u \|^2 = ((u,u)), \quad \forall u, v \in V.$$

$|\cdot|$, $(\cdot,\cdot)_0$ denote the norm and inner product of H respectively.

As is well known, A is a positive unbounded operator in H with a compact inverse, and it has eigenvalues $\{\lambda_j\}$ and eigenvectors $\{\omega_j\}$ satisfying:

$$A\omega_j = \lambda_j \omega_j, \quad j = 1, 2, \cdots,$$
$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots,$$
$$\lambda_j \to \infty \quad (j \to \infty).$$

So we can define $A^s (s \in R)$ as

$$A^s u = \sum_{j=1}^{\infty} \lambda_j^s (u, \omega_j)_0 \omega_j, \quad \forall u \in D(A^s).$$

Let

$$V_{2s} = D(A^s),$$
$$(u,v)_{2s} = \sum_{j=1}^{\infty} \lambda_j^{2s} (u,\omega_j)_0 (v,\omega_j)_0,$$
$$| u |_{2s}^2 = (u,u)_{2s}, \quad \forall u, v \in V_{2s}.$$

* Supported by the Major State Basic Research Development Program of China (G1999032801) and the National Natural Science Foundation of China (Grant Nos. 10001028, 10101020)

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Especially,

$$V = D(A^{\frac{1}{2}}), \quad ((\cdot,\cdot)) = (\cdot,\cdot)_1, \quad \|\cdot\| = |\cdot|_1.$$

For any Banach space E, $BC(R,E)$ is the Banach space consisting of all bounded continuous functions from R to E with the norm $|\cdot|_{BC(R,E)} = \max_{t\in R}|\cdot|_E$, $|\cdot|_E$ is the norm in E.

Let $T(h)$ be a translation operator along time-axis defined as

$$T(h)f(x,t) = f(x,t+h).$$

From now on, we make the following

Assumptions 1

(i) $f(x,t)$ is almost periodic or quasiperiodic in time t,

(ii) $f(x,t)\in BC(R,H)$, $\dfrac{\partial f(x,t)}{\partial t}\in BC(R,H)$,

(iii) there exists $G\in C^2(V;R)$, $G(0)=0$ and $p\in C^1(V;H)$ such that $g(\phi)=G'(\phi)+p(\phi)$, $\forall \phi\in V$ and G (resp. p) is bounded from V into R (resp. H) satisfying

$$\liminf_{\|\phi\|\to\infty}\frac{G(\phi)}{\|\phi\|^2}\geqslant 0,$$

and there exists $c_1>0$, $c_2>0$, $\gamma_1>0$ such that

$$\lim_{\|\phi\|\to\infty}\frac{(\phi,g(\phi))_0-c_1G(\phi)}{\|\phi\|^2}\geqslant 0,$$

$$|p(\phi)|\leqslant c_2(1+|G(\phi)|)^{\frac{1}{2}-\gamma_1}, \quad \forall \phi\in V.$$

(iv) g is a C^1-bounded operator from V into H, Frechet differentiable with differential g'. g maps $D(A)$ into V and is Lipschitzian from the bounded sets of $D(A)$ into V and from the bounded sets of V into H. There exists $\gamma_2>0$, $\gamma_3\in R$, $c_3>0$ and for every $\tilde{R}\geqslant 0$, there exists $c_0=c_0(\tilde{R})$ such that

$$\|g(\phi)\|\leqslant c_0(\tilde{R})(1+|A\phi|)^{1-\gamma_2},$$
$$\forall \phi\in D(A), \quad \|\phi\|\leqslant\tilde{R}$$
$$|g(\phi)-g(\varphi)|$$
$$\leqslant c_3(1+\|\phi\|+\|\varphi\|)^{\gamma_3}\|\phi-\varphi\|,$$
$$\forall \phi,\varphi\in V.$$

(v) g' is a bounded continuous mapping from V to $\mathscr{L}(V,H)$ and a bounded mapping from $D(A)$ into $\mathscr{L}(V_s,H)$ for some $s\in[0,1)$ and from V to $\mathscr{L}(H,V_{-1+\gamma_4})$ for some $0<\gamma_4\leqslant 1$.

Let

$$\Sigma = \text{closure of } \{T(h)f(x,t)=f(x,t+h),\ h\in R\} \text{ in } BC(R,H).$$

Definition 1. A two-parametric family of operators $\{U_\sigma(t,\tau), t\geqslant\tau, \tau\in R\}(\sigma\in\Sigma)$: $E\to E$ are called processes if:

(i) $\lim\limits_{t\to\tau}U_\sigma(t,\tau)v=v$, $\forall v\in E$,

(ii) $U_\sigma(t,s)\cdot U_\sigma(s,\tau)=U_\sigma(t,\tau)$, $\forall t\geqslant s\geqslant\tau$, $\tau\in R$,

(iii) $U_\sigma(\tau,\tau)=I$, I is the identity operator in E, $\tau\in R$.

Definition 2. A set P belonging to a Banach space E is said to be a uniformly attracting set for a family of processes $\{U_\sigma(t,\tau)\}(\sigma\in\Sigma)$: $E\to E$ with respect to Σ if for any bounded set $B\in E$,

$$\lim_{t\to\infty}\sup_{\sigma\in\Sigma}\text{dist}_E(U_\sigma(t,\tau)B,P)=0, \quad \forall\tau\in R.$$

Lemma 1. Under Assumptions 1, $\forall(u_{0\tau},u_{1\tau})\in E_0$, there exists a unique solution $u(x,t)$ to Eq. (1) satisfying $u(x,t)\in BC([\tau,\infty);V)$, $u'(x,t)\in BC([\tau,\infty);H)$. Moreover, if $(u_{0\tau},u_{1\tau})\in E_1$, then $u(x,t)\in BC([\tau,\infty);D(A))$, $u'(x,t)\in BC([\tau,\infty);V)$.

This lemma can be proved by use of Faedo-Galerkin method[1]. We omit the proof here.

From this proposition, we can define a family of processes $\{U_\sigma(t,\tau)\}(\sigma\in\Sigma)$ such that

$$(u(x,t),u'(x,t))=U_\sigma(t,\tau)(u_{0\tau},u_{1\tau}).$$

Lemma 2. The family of processes $\{U_\sigma(t,\tau)\}(\sigma\in\Sigma)$ have a uniformly attracting set

$$\mathscr{D}=\{(\tilde{u},\tilde{v})\in E_0:\ \|\tilde{u}\|^2+|\tilde{v}|^2\leqslant\rho_0^2\},$$

where $\rho_0^2=\dfrac{4}{3}(1+\varepsilon\lambda_1^{-\frac{1}{2}})^2\left(c_3'+\dfrac{4}{\alpha}|f|^2_{BC(R,H)}\right)$, $c_3'>0$ is a constant.

Proof. Let $\varepsilon_0=\min\left(\dfrac{\alpha}{2},\dfrac{\lambda_1}{2\alpha}\right)$, $0<\varepsilon\leqslant\varepsilon_0$, $v=u'+\varepsilon u$, then Eq. (1) can be written as

$$\begin{cases} u'+\varepsilon u-v=0, \\ v'+(\alpha-\varepsilon)v+(A-\varepsilon(\alpha-\varepsilon))u+g(u) \\ \quad =f(x,t). \end{cases} \tag{2}$$

From (iii) of Assumptions 1, there exist two constants k_1, k_2 such that

$$\begin{cases} G(\phi) + \dfrac{1}{8(1+c_1)} \|\phi\|^2 + k_1 \geqslant 0, \\ (\phi, g(\phi))_0 - c_1 G(\phi) + \dfrac{1}{8}\|\phi\|^2 + k_2 \geqslant 0, \\ \quad \forall \phi \in V. \end{cases} \tag{3}$$

Taking scalar product of V with u on both sides of the first equation in (2) and scalar product of H with v on both sides of the first equation in (2) and summing resulted equations together yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + |v|^2) + \varepsilon\|u\|^2 + (\alpha-\varepsilon)|v|^2 - \varepsilon(\alpha-\varepsilon)(u,v)_0 + (g(u),v)_0 = (f,v)_0.$$

Noticing

$$\varepsilon\|u\|^2 + (\alpha-\varepsilon)|v|^2 - \varepsilon(\alpha-\varepsilon)(u,v)_0 \geqslant \frac{\varepsilon}{2}\|u\|^2 + \frac{\alpha}{2}|v|^2, \tag{4}$$

and

$$\begin{aligned}(g(u),v)_0 &= (G'(u)+p(u),u')_0 + \varepsilon(g(u),u)_0 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}G(u) + (p(u),v)_0 - \varepsilon(p(u),u)_0 + \varepsilon(g(u),u)_0 \\ &\geqslant \frac{\mathrm{d}}{\mathrm{d}t}G(u) - c_2(1+|G(u)|)^{\frac{1}{2}-\gamma_1}(|v|+\varepsilon|u|) + \varepsilon c_1 G(u) - \frac{\varepsilon}{8}\|u\|^2 - \varepsilon k_2 \\ &\geqslant \frac{\mathrm{d}}{\mathrm{d}t}G(u) - c_2(1+|G(u)|)^{\frac{1}{2}-\gamma_1}\cdot(|v|+\varepsilon|u|) + \varepsilon c_1 G(u) - \frac{\varepsilon}{4}\|u\|^2 - \varepsilon(k_2+c_1k_1),\end{aligned}$$

we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + |v|^2 + 2G(u)) + \frac{\varepsilon}{4}\|u\|^2 + \frac{\alpha}{2}|v|^2 + \varepsilon c_1 G(u) \\ &\leqslant \varepsilon(k_2+c_1k_1) + (f,v)_0 + c_2(1+G(u))^{\frac{1}{2}-\gamma_1}(|v| + \varepsilon|u|).\end{aligned} \tag{5}$$

On the other hand

$$c_2(1+|G(u)|)^{\frac{1}{2}-\gamma_1}(|v|+\varepsilon|u|) \leqslant \frac{\alpha}{8}|v|^2 + \frac{\varepsilon}{16}\|u\|^2 + c_1'(1+|G(u)|)^{1-2\gamma_1},$$

and by (3),

$$|G(\phi)| \leqslant G(\phi) + \frac{1}{4c_1}\|\phi\|^2 + 2k_1, \quad \forall \phi \in V.$$

Thus

$$\begin{aligned}&c_1'(1+|G(u)|)^{1-2\gamma_1} \\ &\leqslant c_1'\left(1 + 2k_1 + G(u) + \frac{1}{4c_1}\|u\|^2\right)^{1-2\gamma_1} \\ &\leqslant \frac{\varepsilon}{16}\|u\|^2 + \frac{\varepsilon c_1}{4}G(u) + c_2'.\end{aligned}$$

From (5) and above inequalities, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + |v|^2 + 2G(u)) + \frac{\varepsilon}{8}\|u\|^2 + \frac{\alpha}{4}|v|^2 + \frac{3}{4}\varepsilon c_1 G(u) \\ &\leqslant \varepsilon(k_2+c_1k_1) + \frac{2}{\alpha}|f|^2_{BC(R,H)} + c_2'.\end{aligned} \tag{6}$$

Let $Y = \|u\|^2 + |v|^2 + 2c_1G(u) + 2k_1 \geqslant 0$, thus from (6), we deduce that

$$\frac{\mathrm{d}Y}{\mathrm{d}t} + \alpha_2 Y \leqslant c_3' + \frac{4}{\alpha}|f|^2_{BC(R,H)}, \tag{7}$$

where $\alpha_2 = \frac{\varepsilon}{4}$, $c_3' = 2\varepsilon(k_2+c_1k_1) + 2c_2' + 2\alpha_2k_1$.

From (7) and Grownwall inequalities, it follows that

$$\begin{aligned}Y(t) &\leqslant Y(\tau)\exp(-\alpha_2(t-\tau)) + \frac{1}{\alpha_2}\left(c_3' + \frac{4}{\alpha}|f|^2_{BC(R,H)}\right) \\ &\quad\cdot(1-\exp(-\alpha_2(t-\tau))),\end{aligned}$$

where $Y(\tau) = \|u_{0\tau}\|^2 + |u_{1\tau} + \varepsilon u_{0\tau}|^2 + 2c_1 G(u_{0\tau}) + 2k_1$.

Therefore

$$\lim_{t\to\infty} Y(t) \leqslant \frac{1}{\alpha_2}\left(c_3' + \frac{4}{\alpha}|f|^2_{BC(R,H)}\right).$$

Thanks to $Y(t) \geqslant \frac{3}{4}\|u(t)\|^2 + |v|^2$, we get

$$\lim_{t\to\infty}(\|u(t)\|^2 + |v|^2) \leqslant \rho_0^2, \tag{8}$$

where $\rho_0^2 = \frac{4}{3}(1+\varepsilon\lambda_1^{-\frac{1}{2}})^2\left(c_3' + \frac{4}{\alpha}|f|^2_{BC(R,H)}\right)$.

Let

$$\mathscr{D} = \{(\tilde{u},\tilde{v}) \in E_0 : \|\tilde{u}\|^2 + |\tilde{v}|^2 \leqslant \rho_0^2\},$$

then for any bounded set $B \subset E_0$ we have

$$\lim_{t\to\infty}\sup_{\sigma\in\Sigma} \mathrm{dist}_{E_0}(U_\sigma(t,\tau)B, D) = 0, \quad \forall \tau \in R.$$

So system (1) has a uniformly attracting set $\mathscr{D}$. The proof is completed.

For later use, let us give priori-estimates of $u(x,t)$ and $u'(x,t)$ in E_1.

Taking the scalar product of V with Au on both sides of the first equation in (2) and the scalar product of H with Av on both sides of the second equation in (2) and summing resulted equalities together, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|Au|^2+\|v\|^2)+\varepsilon|Au|^2+(\alpha-\varepsilon)\|v\|^2-\varepsilon(\alpha-\varepsilon)(Au,v)_0=(f,Av)_0+((g(u),v)). \quad (9)$$

For $0<\varepsilon\leqslant\varepsilon_0$, we have

$$\varepsilon|Au|^2+(\alpha-\varepsilon)\|v\|^2-\varepsilon(\alpha-\varepsilon)(Au,v)_0\geqslant\frac{\varepsilon}{2}|Au|^2+\frac{\alpha}{2}\|v\|^2$$

and

$$(f,Av)_0=\frac{\mathrm{d}}{\mathrm{d}t}(f,Au)_0-(f',Au)_0+\varepsilon(f,Au)_0\leqslant\frac{\mathrm{d}}{\mathrm{d}t}(f,Au)_0+\frac{\varepsilon}{8}|Au|^2+\frac{4}{\varepsilon}|f'|^2_{BC(R,H)}+4\varepsilon|f|^2_{BC(R,H)}.$$

On the other hand, if $\{u_{0\tau},u_{1\tau}\}$ belongs to a bounded set B of E_1, then B is also bounded in E_0 and there exists $t_0\geqslant\tau$, when $t\geqslant t_0$, $\|u(t)\|^2+|u'(t)|^2\leqslant2\rho_0^2$. From (iv) of Assumptions 1 when $t\geqslant t_0$ we obtain

$$\|g(u(t))\|\leqslant c_0(2\rho_0)(1+|Au(t)|)^{1-\gamma_2}.$$

Thus if $t\geqslant t_0$, it follows that

$$|((g(u),v))|\leqslant c_0(2\rho_0)(1+|Au|)^{1-\gamma_2}\|v\|\leqslant\frac{\alpha}{4}\|v\|^2+\frac{\varepsilon}{8}|Au|^2+c_0'.$$

From (9) and above inequalities, let $\alpha_1=\frac{\varepsilon}{2}$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}(|Au|^2+\|v\|^2-2(f,Au)_0)+\alpha_1(|Au|^2+\|v\|^2)\leqslant2c_0'+8\varepsilon|f|^2_{BC(R,H)}+\frac{8}{\varepsilon}|f'|^2_{BC(R,H)}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(|Au-f|^2+\|v\|^2)+\alpha_1(|Au-f|^2+\|v\|^2)\leqslant2c_0'+(8\varepsilon+\alpha_1)|f|^2_{BC(R,H)}+\frac{8}{\varepsilon}|f'|^2_{BC(R,H)}+2|f|_{BC(R,H)}|f'|_{BC(R,H)}.$$

The Grownwall inequalities imply

$$|Au(t)-f(t)|^2+\|v(t)\|^2\leqslant(|Au(t_0)-f(t_0)|^2+\|v(t_0)\|^2)\cdot\exp(-\alpha_1(t-t_0))+\Big(2c_0'+(8\varepsilon+\alpha_1)|f|^2_{BC(R,H)}+\frac{8}{\varepsilon}|f'|^2_{BC(R,H)}+2|f|_{BC(R,H)}|f'|_{BC(R,H)}\Big)\cdot(1-\exp(-\alpha_1(t-t_0))).$$

So

$$\lim_{t\to\infty}(|Au(t)-f(t)|^2+\|u'(t)\|^2)\leqslant\rho_1^2,$$

where

$$\rho_1^2=(1+\varepsilon\lambda_1^{-\frac{1}{2}})^2\cdot\Big(2c_0'+(8\varepsilon+\alpha_1)|f|^2_{BC(R,H)}+\frac{8}{\varepsilon}|f'|^2_{BC(R,H)}+2|f|_{BC(R,H)}|f'|_{BC(R,H)}\Big).$$

Therefore,

$$\lim_{t\to\infty}(|Au(t)|^2+\|u(t)\|^2)\leqslant\rho_2^2, \quad (10)$$

where $\rho_2^2=3\rho_1^2+2|f|^2_{BC(R,H)}$.

Next we consider the following systems:

$$\begin{cases}\dfrac{\partial^2\hat{u}}{\partial^2t}+\alpha\dfrac{\partial\hat{u}}{\partial t}-\Delta\hat{u}=0, \text{ in } \Omega\times[\tau,\infty),\\ u(x,\tau)=u_{0\tau}(x),\\ \hat{u}'(x,\tau)=u_{1\tau}(x),\\ \hat{u}|_{\partial\Omega}=0, \text{ on } \partial\Omega\times[\tau,\infty),\end{cases} \quad (11)$$

$$\begin{cases}\dfrac{\partial^2\bar{u}}{\partial^2t}+\alpha\dfrac{\partial\bar{u}}{\partial t}-\Delta\bar{u}=f(x,t)-g(u),\\ \quad \text{in } \Omega\times[\tau,\infty),\\ \bar{u}(x,\tau)=\bar{u}'(x,\tau)=0,\\ \bar{u}|_{\partial\Omega}=0, \text{ on } \partial\Omega\times[\tau,\infty).\end{cases} \quad (12)$$

Let $\hat{v}=\hat{u}'+\varepsilon\hat{u}$, and due to (4), we obtain

$$\|\hat{u}(t)\|^2+|\hat{v}(t)|^2\leqslant(\|u_{0\tau}\|^2+|u_{1\tau}+\varepsilon u_{0\tau}|^2)\cdot\exp(-\alpha_1(t-\tau)). \quad (13)$$

Taking partial derivative with respect to t on both sides of the first equation in (12) and denoting $\tilde{w}=\bar{u}'$ yields

$$\begin{cases}\dfrac{\partial^2\tilde{w}}{\partial^2t}+\alpha\dfrac{\partial\tilde{w}}{\partial t}-\Delta\tilde{w}=f'-g'(u)u',\\ \tilde{w}(x,\tau)=0,\\ \tilde{w}'(x,\tau)=f(x,\tau)-g(u_{0\tau}).\end{cases} \quad (14)$$

Recall (v) of Assumptions 1, we know $f'-g'(u)u'\in V_{-1+\gamma_4}$. For any $\{u_{0\tau},u_{1\tau}\}\in B$, where B is any bounded set in E_0, from Theorem 2, it follows that there exists $t_0(B)\geqslant\tau$, $\|u(t)\|^2+|u'(t)|^2\leqslant2\rho_0^2$ if $t\geqslant t_0(B)$. So $f'-g'(u)u'\in BC([\tau,\infty);V_{-1+\gamma_4})$. From (iv) of Assumptions 1, we get $f(x,\tau)-g(u_{0\tau})\in H$. Similar to Lemma 1 and from the following estimates, system (14) has a unique solution $\tilde{w}(x,t) \in BC([\tau,\infty); V_{\gamma_4})$, $\tilde{w}'(x,t) \in BC([\tau,\infty); V_{-1+\gamma_4})$.

Letting $\overline{w} = \tilde{w}' + \varepsilon\tilde{w}$ and taking scalar product of H with $A^{\gamma_4 - 1}\overline{w}$ on both sides of the first equation in (14), we obtain

$$\frac{d}{dt}(|\tilde{w}|^2_{\gamma_4} + |\overline{w}|^2_{\gamma_4-1}) + \alpha_1(|\tilde{w}|^2_{\gamma_4} + |\overline{w}|^2_{\gamma_4-1}) \leqslant |f'|^2_{BC(R,H)} + |g'(u)|^2_{\mathscr{L}(H,V_{\gamma_4-1})}|u'|^2.$$

If $t \geqslant t_0(B)$ holds, from above we know

$$|f'|^2_{BC(R,H)} + |g'(u)|^2_{\mathscr{L}(H,V_{-1+\gamma_4})}|u'|^2 \leqslant |f'|^2_{BC(R,H)} + 4\rho_0^2 c_4^2(2\rho_0),$$

where $c_4(2\rho_0)$ is a bounded function of $2\rho_0$.

So

$$\begin{aligned} |\tilde{w}(t)|^2_{\gamma_4} + |\overline{w}(t)|^2_{\gamma_4-1} &\leqslant (|\tilde{w}(t_0)|^2_{\gamma_4} + |\overline{w}(t_0)|^2_{\gamma_4-1}) \\ &\quad \cdot \exp(-\alpha_1(t-t_0)) \\ &\quad + (|f'|^2_{BC(R,H)} + 4\rho_0^2 c_4(2\rho_0)) \\ &\quad \cdot (1 - \exp(-\alpha_1(t-t_0))). \end{aligned} \tag{15}$$

Then

$$\lim_{t\to\infty}(\tilde{w}(t)|^2_{\gamma_4} + |\tilde{w}'(t)|^2_{\gamma_4-1}) \leqslant 4\rho_0^2(1+\varepsilon\lambda_1^{-\frac{1}{2}})^2 c_4(2\rho_0) + |f'|^2_{BC(R,H)}. \tag{16}$$

Due to the first equation in (12) and (iv) of Assumptions 1, it follows that

$$\lim_{t\to\infty}|\bar{u}(t)|^2_{1+\gamma_4} + |\bar{u}'(t)|^2_{\gamma_4} \leqslant c_5, \tag{17}$$

where c_5 is a constant depending on α, ρ_0 and $|f'|_{BC(R,H)}$.

2 Almost periodic case

In this section we prove the existence of uniform attractor of system (1) when $f(x,t)$ is almost periodic in time t.

Denote $\Sigma_1 = \Sigma$. In view of definition of Σ, we have

$$|h|_{BC(R,H)} = |f|_{BC(R,H)},$$
$$\left|\frac{\partial h}{\partial t}\right|_{BC(R,H)} = \left|\frac{\partial f}{\partial t}\right|_{BC(R,H)}, \quad \forall h \in \Sigma_2. \tag{18}$$

From Lemma 1, we can define a family of processes $\{U_{\sigma_1}(t,\tau)\}(\sigma_1 \in \Sigma_1)$ and a semigroup $\bar{S}(t-\tau)$ as the following:

$$(u(t), u'(t)) = U_{\sigma_1}(t,\tau)(u_{0\tau}, u_{1\tau}),$$
$$(\hat{u}(t), \hat{u}'(t)) = \bar{S}(t-\tau)(u_{0\tau}, u_{1\tau}),$$

and a family of operators $\{\overline{U}_{\sigma_1}(t,\tau)\}(\sigma_1 \in \Sigma_1)$ such that

$$(\bar{u}(t), \bar{u}'(t)) = \overline{U}_{\sigma_1}(t,\tau)(u_{0\tau}, u_{1\tau}),$$

then

$$U_{\sigma_1}(t,\tau) = \bar{S}(t-\tau) + \overline{U}_{\sigma_1}(t,\tau).$$

Notice that Σ_1 is a compact metric subspace of $BC(R,H)$ under the norm $|\cdot|_{BC(R,H)}$[10].

Definition 3. A closed set $A_{\sigma_1} \subset E_0$ is said to be the uniform attractor of the processes $\{U_{\sigma_1}(t,\tau)\}$ $(\sigma_1 \in \Sigma_1)$, with respect to Σ_1 if it is uniformly attracting with respect to Σ_1, and it is contained in any closed uniformly attracting set A' of the processes $\{U_{\sigma_1}(t,\tau)\}(\sigma_1 \in \Sigma_1)$.

Let us introduce the Kuratowski-measure $\kappa(\Xi)$ of any set Ξ in a Banach space E.

$\kappa(\Xi) = \inf\{r$: there is a finite covering of Ξ by balls of radius less than r in $E\}$

Then, $\kappa(\Xi) = 0$ if and only if Ξ is compact in E.

Theorem 1. The processes $\{U_{\sigma_1}(t,\tau)\}$ are uniformly asymptotically compact in E_0 with respect to Σ_1 as $t \to \infty$.

Proof. For any bounded set $B \subset E_0$, $\overline{U}_{\sigma_1}(t,\tau)\cdot B$ is compact in E_0 because the imbedding $V_{1+\gamma_4} \times V_{\gamma_4} \hookrightarrow E_0$ is compact and inequality (17) holds.

So

$$\kappa(\overline{U}_{\sigma_1}(t,\tau)B) = 0,$$

uniformly with respect to Σ_1.

From (13) $\kappa(\bar{S}(t-\tau)B) \to 0$ uniformly with respect to Σ_1, as $t \to \infty$. In addition,

$$\kappa(U_{\sigma_1}(t,\tau)B) \leqslant \kappa(\bar{S}(t-\tau)B) + \kappa(\overline{U}_{\sigma_1}(t,\tau)B).$$

Then $\kappa(U_{\sigma_1}(t,\tau)B) \to 0$ uniformly with respect to Σ_1, as $t \to \infty$. So the processes $\{U_{\sigma_1}(t,\tau)\}$ are uniformly asymptotically compact in E_0 with respect to Σ_1.

Lemma 3.[9] Let a family of processes $\{U_\sigma(t,\tau)\}(\sigma \in \Sigma)$ acting in the Banach space E be uniformly asymptotically compact with respect to Σ and

$(E\times\Sigma, E)$-continuous. Also let Σ be a compact metric space, $T(t)$ a continuous invariant ($T(t)\Sigma=\Sigma$) semigroup on Σ satisfying $U_\sigma(t+s,\tau+s)=U_{T(s)\sigma}(t,\tau)(s\geqslant 0)$, then the semigroup $S(t)(u,\sigma)=(U_\sigma(t,0)u, T(t)\sigma)$, $\forall t\geqslant 0$, $(u,\sigma)\in E\times\Sigma$ possesses a compact attractor A satisfying $S(t)A=A(\forall t\geqslant 0)$, moreover, $\prod A=A_\Sigma$ is the uniform attractor of the family of processes $\{U_\sigma(t,\tau)\}$ with respect to Σ, where $\prod:(u,\sigma)=u$ is a projecting operator.

Theorem 2. When $f(x,t)$ is almost periodic in time t and satisfies Assumptions 1, then the semigroup

$$S(t)(u,\sigma_1)=(U_{\sigma_1}(t,0)u, T(t)\sigma_1)$$
$$(u,\sigma_1)\in E_0\times\Sigma_1,\quad t\geqslant 0$$

possesses a compact attractor A_1 in $E_0\times BC(R,H)$. Moreover, $A_{\Sigma_1}=\prod A_1$ is the uniform attractor of the family of processes $\{U_{\sigma_1}(t,\tau)\}$, where $\prod$: $(u,\sigma_1)=u$, $\forall(u,\sigma_1)\in E_0\times\Sigma_1$ is a projecting operator.

Proof. From equality (18) and Theorem 1, we know the processes $\{U_{\sigma_1}(t,\tau)\}$ are uniformly asymptotically compact with respect to Σ_1. Σ_1 is a compact subspace of $BC(R,H)$ under the norm $|\cdot|_{BC(R,H)}$. From Lemma 1, $\forall\sigma_1\in\Sigma_1$, $U_{\sigma_1}(t+s,\tau+s)=U_{T(s)\sigma_1}(t,\tau)(s\geqslant 0)$ holds. Hence according to Lemma 3, we only need prove the processes $\{U_{\sigma_1}(t,\tau)\}$ are $(E_0\times\Sigma_1, E_0)$-continuous.

Let $u_1(x,t)$, $u_2(x,t)$ be the solutions of the first equation in (1) with corresponding external force terms $f_1(x,t)$, $f_2(x,t)$ and initial data $(u_{01\tau},u_{11\tau})$, $(u_{02\tau},u_{12\tau})\in E_0$.

Denote $q=u_1(x,t)-u_2(x,t)$, $\bar f(x,t)=f_1(x,t)-f_2(x,t)$, then q satisfies the following equations:

$$\begin{cases}\dfrac{\partial^2 q}{\partial t^2}+\alpha\dfrac{\partial q}{\partial t}-\Delta q+(g(u_1)-g(u_2))=\bar f(x,t),\\ q(x,\tau)=u_{01\tau}-u_{02\tau},\quad q'(x,\tau)=u_{11\tau}-u_{12\tau}.\end{cases}\tag{19}$$

Taking scalar product of H with q' on both sides of the first equation in (19) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|q\|^2+|q'|^2)+\alpha|q'|^2=-(g(u_1)-g(u_2),q')_0+(\bar f,q')_0.\tag{20}$$

Recall (v) of Assumptions 1, we know

$$|g(u_1(t))-g(u_2(t))|\leqslant c_3(1+\|u_1(t)\|^2+\|u_2(t)\|^2)^{\gamma_3}\|u_1(t)-u_2(t)\|.$$

From (20), Lemma 2 and the above inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|q\|^2+|q'|^2)\leqslant\hat c_3(t)(\|q\|^2+|q'|^2)+\frac{4}{\sigma}|\bar f|^2,\tag{21}$$

where $\hat c_3(t)$ is a bounded function of t.

Thus from (21), it follows that

$$\begin{aligned}\|q(t)\|^2+|q'(t)|^2&\leqslant(\|q(\tau)\|^2+|q'(\tau)|^2)\\&\quad\cdot\exp(\max_{s\in[\tau,t]}\hat c_3(s)(t-\tau))\\&\quad+\int_\tau^t|\bar f(s)|^2\mathrm{d}s(\max_{s\in[\tau,t]}\hat c_3(s)(t-\tau)),\end{aligned}$$

i.e.

$$\begin{aligned}\|u_1(t)-u_2(t)\|^2+|u_1'(t)-u_2'(t)|^2&\leqslant(\|u_{01\tau}-u_{02\tau}\|^2+|u_{11\tau}-u_{12\tau}|^2\\&\quad+(t-\tau)|f_1-f_2|^2_{BC(R,H)})\\&\quad\cdot\exp(\max_{s\in[\tau,t]}\hat c_3(s)(t-\tau)).\end{aligned}$$

Then for any fixed time $t\geqslant\tau$, $\|u_1(t)-u_2(t)\|\to 0$ and $|u_1'(t)-u_2'(t)|\to 0$ when $\|u_{01\tau}-u_{02\tau}\|$, $|u_{11\tau}-u_{12\tau}|\to 0$ and $|f_1-f_2|_{BC(R,H)}\to 0$. So the processes $\{U_{\sigma_1}(t,\tau)\}$ are $(E_0\times\Sigma_1,E_0)$-continuous. The theorem has been proved.

3 Quasiperiodic case

Assume that $f(x,t)$ is quasiperiodic in time t, hence, there exist a group of rationally independent real numbers $\bar\alpha_1,\cdots,\bar\alpha_k$ satisfying

$$\begin{aligned}f(x,\bar\alpha_1 t,\cdots,\bar\alpha_i t+2\pi,\cdots,\bar\alpha_k t)&=f(x,\bar\alpha_1 t,\cdots,\bar\alpha_i t,\cdots,\bar\alpha_k t)\\&\quad(1\leqslant i\leqslant k).\end{aligned}$$

Denote

$$\begin{aligned}\bar\alpha t&=(\bar\alpha_1 t,\cdots,\bar\alpha_k t),\quad\bar\alpha=(\bar\alpha_1,\cdots,\bar\alpha_k),\\ w(t)&=(w_1(t),\cdots,w_k(t))=[\bar\alpha t+w_0]\\&=(\bar\alpha t+w_0)\mathrm{mod}(2\pi)^k,\\ w_0&=(w_{01},\cdots,w_{0k})\in T^k=[0,2\pi]^k,\\ F(x,w(t))&=f(x,t).\end{aligned}$$

In addition to Assumptions 1, we make the following

Assumptions 2.

$$\frac{\partial F(x, w(t))}{\partial w_i} \in BC(T^k, H), \quad (i = 1, \cdots, k).$$

Let $\Sigma_2 = \Sigma$, by virtue of definition of Σ, it is clear that

$$|h|_{BC(R,H)} = |f|_{BC(R,H)},$$

$$\left|\frac{\partial h}{\partial t}\right|_{BC(R,H)} = \left|\frac{\partial f}{\partial t}\right|_{BC(R,H)}, \quad \forall h \in \Sigma_2. \quad (22)$$

Moreover, there exists $w^* \in T^k$ satisfying

$$h(x, t) = f(x, \bar{\alpha} t + w^*). \quad (23)$$

From Lemma 1, we can define a family of processes $U_{\sigma_2}(t, \tau)(u_{0\tau}, u_{1\tau}) = (u(x, t), u'(x, t))(\sigma_2 \in \Sigma_2)$. But from (23), the processes $U_{\sigma_2}(t, \tau)$ can be transformed into the processes $U_{\bar{w}}(t, \tau)(\bar{w} \in T^k)$ such that $U_{\bar{w}}(t, \tau)(u_{0\tau}, u_{1\tau}) = (u(x, t), u'(x, t))$.

$\forall \bar{w} \in T^k$, $T_1(t)\bar{w} = [\bar{\alpha} t + \bar{w}]$, then $T_1(t)$ is a continuous invariant semigroup acting on T^k.

We can define the semigroup $S(t)$ acting on $E_0 \times T^k$ as the following:

$$S(t)(u, \bar{w}) = (U_{\bar{w}}(t, 0)u, T_1(t)\bar{w}),$$
$$\forall (u, \bar{w}) \in E_0 \times T^k.$$

Because T^k is a compact set of R^k, according to Lemma 3, in order to prove the existence of uniform attractor we only need prove $\{U_{\bar{w}}(t, \tau)\}$ are $(E_0 \times T^k, E_0)$-continuous. If $w_1, w_2 \in T^k$, $|w_1 - w_2| \to 0$, then from Assumptions 1, $|f(x, \bar{\alpha} t + w_1) - f(\alpha t + w_2)|_{BC(R,H)} \to 0$. Therefore, by a similar argument of Theorem 2, we have

Theorem 3. When $f(x, t)$ is quasiperiodic in time t and satisfies Assumptions 1, the semigroup $S(t)(u, \bar{w}) = (U_{\bar{w}}(t, 0)u, T_1(t)\bar{w})$ possesses a compact attractor A_2 in $E_0 \times R^k$. Moreover, $\prod A_2 = A_T$ is the uniform attractor of processes $\{U_{\bar{w}}(t, \tau)\}$, $\bar{w} \in T^k$. $\forall (u, \bar{w}) \in E_0 \times T^k$, $\prod : (u, \bar{w}) = u$ is a projecting operator.

Now we give the upper bounds of Hausdorff and Fractal dimensions of A_T.

Theorem 4. The Hausdorff and Fractal dimensions of uniform attractor A_T in Theorem 3 satisfy the following inequalities:

$$d_H(A_T) \leqslant d_H(A_2) \leqslant k + \hat{m},$$
$$d_F(A_T) \leqslant d_F(A_2) \leqslant 2k + 2\hat{m},$$

where $\hat{m}$ satisfies the following inequalities,

$$\hat{m} \geqslant \begin{cases} 2^8 \dfrac{G^2}{\alpha^2}\left(1 + \dfrac{\alpha^2}{\lambda_1}\right)^2 k + 2^7 \dfrac{\gamma^2}{\alpha^2}\left(1 + \dfrac{\alpha^2}{\lambda_1}\right)^2 \hat{c}\lambda_1 \ln \hat{m} \\ \quad (n = 2 \text{ and } s = 0), \\ 2^8 \dfrac{G^2}{\alpha^2}\left(1 + \dfrac{\alpha^2}{\lambda_1}\right)^2 k + 2^7 \dfrac{\gamma^2}{\alpha^2}\left(1 + \dfrac{\alpha^2}{\lambda_1}\right) \hat{m}^{1 - \frac{2(1-s)}{n}} \\ \quad (n \neq 2 \text{ or } s \neq 0). \end{cases}$$

Proof. First, we transform the system (1) into the following autonomous system through the semigroup $S(t)(u_0, w_0) = (U_{w_0}(t, 0)u_0, T_1(t)w_0)$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \alpha \dfrac{\partial u}{\partial t} - \Delta u + g(u) = F(x, w(t)), \\ \dfrac{dw(t)}{dt} = \bar{\alpha} \\ (u(t), u'(t))|_{t=0} = (u_{01}, u_{02}) = u_0 \in E_0, \\ \quad w(t)|_{t=0} = w_0 \in T^k. \end{cases} \quad (24)$$

Then, let

$$y(t) = (u(x, t), v(x, t), w(t)),$$
$$M(y(t)) = (v - \varepsilon u, -(\alpha - \varepsilon)v - (A - \varepsilon(\alpha - \varepsilon))u - g(u) + F(x, w(t)), \bar{\alpha})^T,$$

where $v(x, t) = u'(x, t) + \varepsilon u(x, t)$.

Thus the system (24) can be rewritten as

$$\begin{cases} \dfrac{\partial y(t)}{\partial t} = M(y(t)), \\ y(t)|_{t=0} = y_0 = (u_{01}, v_0, w_0), \end{cases} \quad (25)$$

where $v_0 = u_{02} + \varepsilon u_{01}$.

Let $y(t) = (u(t), v(t), w(t))^T$ be the solution of system (24) with initial value $y_0 \in A_2$. From Lemma IV.3.5[1], $(u(t), v(t)) \in E_1$ and satifies (10). The linearized equation of (25) at $y(t)$ is

$$\begin{cases} \dfrac{\partial z(t)}{\partial t} = M'(y(t))z(t), \\ z(t)|_{t=0} = z_0. \end{cases} \quad (26)$$

In Eq. (25), $z(t) = (z_1(t), z_2(t), \mu(t))^T$, $\mu(t) = (\mu_1(t), \cdots, \mu_k(t))$, $z_0 = (z_{01}, z_{02}, \mu_0) \in E_0 \times T^k$, $M'(y(t))z(t) = (z_2(t) - \varepsilon z_1(t), -(\alpha - \varepsilon)z_2(t) - (A - \varepsilon(\alpha - \varepsilon)I)z_1(t) - g'(u) \cdot z_1(t) + F'_u \mu, 0)^T$.

For simplicity, we introduce some notations as the following:

$$((\xi, \zeta), (\bar{\xi}, \bar{\zeta}))_{E_0} = ((\xi, \bar{\xi})) + (\zeta, \bar{\zeta})_2,$$
$$\forall (\xi, \zeta), (\bar{\xi}, \bar{\zeta}) \in E_0,$$
$$\|(v_1, \mu_1)\|^2_{E_0 \times R^k} = ((v_1, \mu_1), (v_1, \mu_1)),$$
$$((v_1, \mu_1), (v_2, \mu_2)) = (v_1, v_2)_{E_0} + (\mu_1, \mu_2)_{R^k},$$

$$\forall (v_1, \mu_1), (v_2, \mu_2) \in E_0 \times R^k,$$

where $(\cdot, \cdot)_{R^k}$ is the inner product in R^k.

In the following let us give the estimate of $(M'(y(t))z, z)$.

At first, we recall that

$$(M'(y(t))z, z) = ((z_2(t) - \varepsilon z_1(t), z_1(t))) + (-(\alpha - \varepsilon)z_2(t) - (A - \varepsilon(\alpha - \varepsilon)I)z_1(t) - g'(u)\cdot z_1(t) + F_w' \cdot \mu, z_2(t)),$$

hence

$$((z_2(t) - \varepsilon z_1(t), z_1(t))) + (-(\alpha - \varepsilon)z_2(t) - (A - \varepsilon(\alpha - \varepsilon)I)z_1(t), z_2(t))_2 \leqslant -\alpha_1(\| z_1(t) \|^2 + | z_2(t) |^2),$$

$$(g'(u) \cdot z_1(t), z_2(t))_2 \leqslant | g'(u)z_1(t) | | z_2(t) | \leqslant \frac{\alpha_1}{2} | z_2(t) |^2 + \frac{1}{2\alpha_1} | g'(u)z_1(t) |^2.$$

From (v) of Assumptions 1 and estimates of $(u(x, t), u'(x, t))$ in E_1, let

$$\gamma = \sup_{v \in D(A), |Av| \leqslant \rho_2} | g'(v) |_{\mathscr{L}(V_s, H)} < \infty.$$

Then,

$$(M'(y(t))z, z) \leqslant -\frac{\alpha_1}{2}(\| z_1(t) \|^2 + | z_2(t) |^2) + \frac{\gamma^2}{2\alpha_1} | z_1(t) |_s^2 + \frac{bG}{2}(z_2(t) |^2 + \frac{G}{2b} | \mu |^2,$$

where b is any positive constant,

$$G = \left(\sum_{i=1}^{k} \left| \frac{\partial F}{\partial w_i} \right|^2_{BC(R, H)} \right)^{\frac{1}{2}}.$$

Denote

$$(M_1(y(t))z, z) = (\overline{M}_1(z_1, z_2), (z_1, z_2))_{E_0} + (\hat{M}_1\mu, \mu)_{R^k},$$

$$\overline{M}_1(z_1, z_2) = \left(-\frac{\alpha_1}{2}z_1(t) + \frac{\gamma^2}{2\alpha_1}A^{-1+s}z_1(t), \left(\frac{G}{2b} - \frac{\alpha_1}{2}\right)z_2(t) \right),$$

$$\hat{M}_1\mu = \frac{G}{2b}I_k\mu,$$

where I_k is the identity operator in R^k. Then the operator $\hat{M}_1 = \begin{pmatrix} \overline{M}_1 & 0 \\ 0 & \hat{M}_1 \end{pmatrix}$ is diagonal. $(z_{11}(t), z_{12}(t), 0), \cdots, (z_{i1}(t), z_{i2}(t), 0), \cdots, (z_{(m-k)1}(t), z_{(m-k)2}(t), 0)$ are the corresponding solutions of Eq. (26) with initial values $(\xi_{11}, \zeta_{12}, 0), \cdots, (\xi_{i1}, \zeta_{i2}, 0), \cdots, (\xi_{(m-k)1}, \zeta_{(m-k)2}, 0)$, where $(\xi_{11}, \zeta_{12}), \cdots, (\xi_{(m-k)1}, \zeta_{(m-k)2})$ are linear independent bases of E_0.

Let $\bar{z}_i = (z_{i1}, z_{i2})(1 \leqslant i \leqslant m-k)$. $\phi_1 = (\varphi_1, \eta_1), \cdots, \phi_{n-k} = (\varphi_{m-k}, \eta_{m-k})$ are orthonormal bases of span$\{\bar{z}_1(t), \cdots, \bar{z}_{m-k}(t)\}$, $\hat{\mu}_{m-k+1}, \cdots, \hat{\mu}_m$ are orthonormal bases of R^k. Then $(\phi_1, 0), \cdots, (\phi_{m-k}, 0)$, $(0, \hat{\mu}_{m-k+1}), \cdots, (0, \hat{\mu}_m)$ are orthonormal in $E_0 \times R^k$. From Theorem 3, the semigroup $S(t)(u, w) = (U_{w_0}(t, 0)u, T_1(t)w_0)$ possesses the compact attractor A_2. Moreover, $\dim A_T \leqslant \dim A_2$.

Denote

$$\theta_i = (\phi_i, 0)(i = 1, \cdots, m-k),$$

$$\theta_i = (0, \hat{\mu}_i)(m - k + 1 \leqslant i \leqslant m),$$

$$q_m = \lim_{T \to \infty} \sup_{\| \theta_i \|_{E_0 \times R^k} \leqslant 1} \sup_{y_0 \in A_2} \cdot \left(\frac{1}{T}\int_0^T \sum_{i=1}^{m} (M'(y(s))\theta_i, \theta_i)\mathrm{d}s \right).$$

In order to give upper bounds of Hausdorff and Fractal dimensions of A_2, we need the following lemma[1]:

Lemma 4. A_2 is the compact attractor of the semigroup $S(t)$, and there exists an integer $N > 0$ satisfying $q_N < 0$, then the Hausdorff dimensions of A_2 satisfy

$$\mathrm{d}_H(A_2) \leqslant N.$$

The Fractal dimensions of A_2 satisfy

$$\mathrm{d}_F(A_2) \leqslant N\left(1 + \max_{1 \leqslant j \leqslant N-1} \frac{(q_j)_+}{| qN |}\right).$$

Since

$$\begin{aligned} &\sum_{i=1}^{m}(M'(y(t))\theta_i, \theta_i) \\ &\leqslant -\frac{\alpha_1}{2}\sum_{i=1}^{m-k}(\| \varphi_i \|^2 + | \eta_i |^2) + \frac{\gamma^2}{2\alpha_1}\sum_{i=1}^{m-k} | \varphi_i |_s^2 + \frac{bG}{2}\sum_{i=1}^{m-k} | \eta_i |^2 + \frac{G}{2b}k \\ &\leqslant -\frac{\alpha_1}{4}\sum_{i=1}^{m-k}(\| \varphi_i \|^2 + | \eta_i |^2) + \frac{\gamma^2}{2\alpha_1}\sum_{i=1}^{m-k} | \varphi_i |_s^2 + \frac{G^2}{\alpha_1}k\left(\text{let} b = \frac{\alpha_1}{2G}\right) \\ &= -\frac{\alpha_1}{4}(m-k) + \frac{G^2}{\alpha_1}k + \frac{\gamma^2}{2\alpha_1}\sum_{i=1}^{m-k} | \varphi_i |_s^2 \\ &\leqslant -\frac{\alpha_1}{4}(m-k) + \frac{G^2}{\alpha_1}k + \frac{\gamma^2}{2\alpha_1}\sum_{j=1}^{m-k}\lambda_j^{s-1} \end{aligned}$$

and (see Ref. [1])

$$\sum_{j=1}^{m-k} \lambda_j^{-1} \leqslant \hat{c}\lambda_1 (m-k)^{1-\frac{2(1-s)}{n}}$$
$$(n \neq 2 \text{ or } s \neq 0),$$
$$\sum_{j=1}^{m-k} \lambda_j^{-1} \leqslant \hat{c}\lambda_1 \ln(m-k) \quad (n = 2 \text{ and } s = 0),$$

where $\hat{c}$ is a positive constant. Then,

$$q_m \leqslant \begin{cases} -\frac{\alpha_1}{4}(m-k) + \frac{G^2}{\alpha_1}k + \frac{\gamma^2}{2\alpha_1}\hat{c}\lambda_1 \ln(m-k) \\ \quad (n=2 \text{ and } s=0), \\ -\frac{\alpha_1}{4}(m-k) + \frac{G^2}{\alpha_1}k + \frac{\gamma^2}{2\alpha_1}\hat{c}\lambda_1 (m-k)^{1-\frac{2(1-s)}{n}} \\ \quad (n \neq 2 \text{ or } s \neq 0). \end{cases}$$

Let $\hat{m} = m - k$ and choose $\hat{m}$ sufficiently large so that,

$$\frac{G^2}{\alpha_1^2}\frac{k}{\hat{m}} + \frac{\gamma^2}{2\alpha_1^2}\hat{c}\lambda_1 \frac{\ln \hat{m}}{\hat{m}} \leqslant \frac{1}{4}$$
$$(n = 2 \text{ and } s = 0), \tag{27}$$
$$\frac{G^2}{\alpha_1^2}\frac{k}{\hat{m}} + \frac{\gamma^2}{2\alpha_1^2}\hat{c}\lambda_1 \hat{m}^{-\frac{2(1-s)}{n}} \leqslant \frac{1}{4}$$
$$(n \neq 2 \text{ or } s \neq 0). \tag{28}$$

It is easy to show that

$$q_m \leqslant -\frac{3}{8}\hat{m}\alpha_1,$$

and for $j = 1, \cdots, m$,

$$(q_j)_+ \leqslant \frac{1}{4}\hat{m}\alpha_1, \quad \max_{1 \leqslant j \leqslant m-1} \frac{(q_j)_+}{|q_m|} \leqslant 1.$$

Choose $\varepsilon = \varepsilon_0$, then $\alpha_1 = \min\left(\frac{\alpha}{8}, \frac{\lambda_1}{4\alpha}\right)$ and

$$\frac{1}{\alpha_1} = \max\left(\frac{8}{\alpha}, \frac{4\alpha}{\lambda_1}\right).$$

We can choose a larger m by replacing $\frac{1}{\alpha_1}$ in (27) and (28) by $\frac{8}{\alpha}(1 + \frac{\alpha^2}{\lambda_1})$, then

$$\hat{m} \geqslant \begin{cases} 2^8 \frac{G^2}{\alpha^2}\left(1 + \frac{\alpha^2}{\lambda_1}\right)^2 k + 2^7 \frac{\gamma^2}{\alpha^2}\left(1 + \frac{\alpha^2}{\lambda_1}\right)^2 \hat{c}\lambda_1 \ln \hat{m} \\ \quad (n=2 \text{ and } s=0), \\ 2^8 \frac{G^2}{\alpha^2}\left(1 + \frac{\alpha^2}{\lambda_1}\right)^2 k + 2^7 \frac{\gamma^2}{\alpha^2}\left(1 + \frac{\alpha^2}{\lambda_1}\right) \hat{m}^{1-\frac{2(1-s)}{n}} \\ \quad (n \neq 2 \text{ or } s \neq 0). \end{cases} \tag{29}$$

Choose suitable $\hat{m}$ satisfying (29), then from Lemma 4, we get

$$d_H(A_T) \leqslant d_H(A_2) \leqslant k + \hat{m},$$
$$d_F(A_T) \leqslant d_F(A_2) \leqslant 2k + 2\hat{m}.$$

This theorem is completed.

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